Linear Algebra I

08/04/2016, Friday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 $(2 + 8 + 5 = 15 \text{ pts})$ **Linear systems of equations**

Consider the system of equations

$$\begin{bmatrix} 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 8 \end{bmatrix}.$$

a. Write down the augmented matrix.

b. Determine all values of a and b for which the system is consistent.

c. Take $a = b = 3$ and find all solutions.

Required Knowledge: **Gauss-elimination, row operations, (reduced) row echelon form.**

Solution:

1a: The augmented matrix is given by:

$$\begin{bmatrix} 0 & 0 & 3 & 6 & \vdots & a \\ 0 & 0 & 1 & 2 & \vdots & b \\ 0 & 2 & 4 & 6 & \vdots & 8 \end{bmatrix}.$$

1b: By applying row operations, we obtain:

$$\begin{bmatrix} 0 & 0 & 3 & 6 & \vdots & a \\ 0 & 0 & 1 & 2 & \vdots & b \\ 0 & 2 & 4 & 6 & \vdots & 8 \end{bmatrix} \xrightarrow[\textbf{3rd} = \textbf{1st}]{\textbf{1st} = \textbf{3rd}} \begin{bmatrix} 0 & 2 & 4 & 6 & \vdots & 8 \\ 0 & 0 & 1 & 2 & \vdots & b \\ 0 & 0 & 3 & 6 & \vdots & a \end{bmatrix} \xrightarrow[\textbf{3rd} = \textbf{3rd} - 3 \times \textbf{2nd}]{\textbf{1st} = \frac{1}{2} \times \textbf{1st}} \begin{bmatrix} 0 & 1 & 2 & 3 & \vdots & 4 \\ 0 & 0 & 1 & 2 & \vdots & b \\ 0 & 0 & 0 & 0 & \vdots & a - 3b \end{bmatrix}$$

Therefore, the system is consistent if and only if $a = 3b$.

1c: There are no solutions as $a \neq 3b$ for $a = b = 3$.

Let A and B be $n \times n$ matrices. Show that

$$\det(\begin{bmatrix} A & B \\ B & A \end{bmatrix}) = \det(A+B)\det(A-B).$$

REQUIRED KNOWLEDGE: **Determinant, partitioned matrices**

SOLUTION:

Note that

$$\begin{bmatrix} I_n & 0_{n,n} \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} A & B \\ B-A & A-B \end{bmatrix}$$

and that

$$\begin{aligned} \begin{bmatrix} I_n & 0_{n,n} \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} I_n & 0_{n,n} \\ I_n & I_n \end{bmatrix} &= \begin{bmatrix} A & B \\ B-A & A-B \end{bmatrix} \begin{bmatrix} I_n & 0_{n,n} \\ I_n & I_n \end{bmatrix} \\ &= \begin{bmatrix} A+B & B \\ 0_{n,n} & A-B \end{bmatrix}. \end{aligned}$$

Therefore, we have

$$\det(\begin{bmatrix} I_n & 0_{n,n} \\ -I_n & I_n \end{bmatrix})\det(\begin{bmatrix} A & B \\ B & A \end{bmatrix})\det(\begin{bmatrix} I_n & 0_{n,n} \\ I_n & I_n \end{bmatrix}) = \det(\begin{bmatrix} A+B & B \\ 0_{n,n} & A-B \end{bmatrix}).$$

This leads to

$$\det(\begin{bmatrix} A & B \\ B & A \end{bmatrix}) = \det(\begin{bmatrix} A+B & B \\ 0_{n,n} & A-B \end{bmatrix})$$

since $\det(\begin{bmatrix} I_n & 0_{n,n} \\ -I_n & I_n \end{bmatrix}) = \det(\begin{bmatrix} I_n & 0_{n,n} \\ I_n & I_n \end{bmatrix}) = 1$. Note that

$$\det(\begin{bmatrix} A & B \\ B & A \end{bmatrix}) = \det(A+B)\det(A-B).$$

3 $(2+3+3+7=15 \text{ pts})$ **Vector spaces**

Consider the vector space P_3.

a. Are the vectors $x, x^2+1, (x+1)^2$ linearly independent?

b. Is the set
$$\{p(x) \in P_3 \mid p(2) = 1\}$$
a subspace of P_3? Justify your answer.

c. Is the set
$$\{p(x) \in P_3 \mid p(2) = 0\}$$
a subspace of P_3? If so, find its dimension.

d. Let $L : P_3 \to P_3$ be given by
$$L\big(p(x)\big) = p(1)x + p(2).$$
Is L a linear transformation? If so, find its matrix representation with respect to the basis $\{1, x, x^2\}$.

REQUIRED KNOWLEDGE: **Vector space, subspace, linear independence, basis, dimension, linear transformation.**

SOLUTION:

3a: Let a, b, c be scalars such that
$$ax + b(x^2+1) + c(x+1)^2 = 0.$$
Then, we get
$$(b+c)x^2 + (a+2c)x + b + c = 0.$$
This leads to
$$\begin{aligned} b + c &= 0 \\ a + 2c &= 0 \\ b + c &= 0. \end{aligned}$$
Note that
$$a = 2 \qquad b = 1 \qquad c = -1$$
is a solution for the equations above. As such, the vectors $x, x^2+1, (x+1)^2$ are not linearly independent.

3b: Let
$$S := \{p(x) \in P_3 \mid p(2) = 1\}.$$
Notice that the zero polynomial does not belong to this set. As such, it cannot be a subspace. Alternative ways to see this include:

- Let $p(x) \in S$ and define $q(x) = 2p(x)$. Note that $q(2) = 2 \cdot 1 = 2$. Then, $q(x) \notin S$.
- Let $p(x)$, $q(x) \in S$. Define $r(x) = p(x) + q(x)$. Note that $r(2) = p(2) + q(2) = 1 + 1 = 2$. Thus, $r(x) \notin S$.

3c: Let
$$S := \{p(x) \in P_3 \mid p(2) = 0\}.$$
As it contains the zero polynomial, S is non-empty. Let
$$p(x) \in S$$

and α be a scalar. This means that

$$p(2) = 0.$$

Note that

$$(\alpha p)(2) = \alpha p(2) = 0.$$

Hence, $(\alpha p)(x) \in S$. So, the set S is closed under scalar multiplication. In order to show that it is closed under vector addition as well, let

$$p(x),\ q(x) \in S.$$

Note that

$$(p+q)(2) = p(2) + q(2) = 0.$$

Therefore, $(p+q)(x) \in S$. Consequently, S is a subspace.

To find its dimension, we need to find a basis for S. Let $p(x) \in S$ be the form

$$p(x) = ax^2 + bx + c.$$

It follows from $p(2) = 0$ that

$$4a + 2b + c = 0.$$

This leads to

$$p(x) = ax^2 + bx - (4a - 2b) = a(x^2 - 4) + b(x - 2).$$

Note that $x^2 - 4$ and $x - 2$ are linearly independent. This means that they form a basis for S. Hence,the dimension of S is 2.

3d: Note that

$$L\big(\alpha p(x)\big) = \alpha p(1)x + \alpha p(2) = \alpha\big(p(1)x + p(2)\big) = \alpha L\big(p(x)\big)$$

and

$$L\big(p(x)+q(x)\big) = \big(p(1)+q(1)\big)x + \big(p(2)+q(2)\big) = p(1)x+p(2)+q(1)x+q(2) = L\big(p(x)\big)+L\big(q(x)\big).$$

Therefore, L is a linear transformation.

In order to find the matrix representation, we first apply L to the basis vector:

$$\begin{aligned} L(1) &= x + 1 = 1\cdot 1 + 1\cdot x + 0\cdot x^2 \\ L(x) &= x + 2 = 2\cdot 1 + 1\cdot x + 0\cdot x^2 \\ L(x^2) &= x + 4 = 4\cdot 1 + 1\cdot x + 0\cdot x^2. \end{aligned}$$

This results in the following matrix representation:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find the best parabola of the form $y = ax^2 + bx + c$ fitting the points:

x	-2	-1	1	2
y	2	$\frac{1}{2}$	$\frac{3}{2}$	2

Required Knowledge: **Least squares problem, normal equations.**

Solution:

We can formulate the following least squares problem by using the given data:

$$\begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \\ 2 \end{bmatrix}.$$

The normal equations for this least squares problem are given by

$$A^T A x = A^T b$$

where

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \\ 2 \end{bmatrix}.$$

Note that

$$A^T A = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{bmatrix} \quad A^T b = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 1 \\ 6 \end{bmatrix}.$$

Then, we can find the inverse of $A^T A$ as

$$(A^T A)^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & -\frac{5}{18} \\ 0 & \frac{1}{10} & 0 \\ -\frac{5}{18} & 0 & \frac{17}{18} \end{bmatrix}.$$

This leads to

$$x = \begin{bmatrix} \frac{1}{9} & 0 & -\frac{5}{18} \\ 0 & \frac{1}{10} & 0 \\ -\frac{5}{18} & 0 & \frac{17}{18} \end{bmatrix} \begin{bmatrix} 18 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{10} \\ \frac{2}{3} \end{bmatrix}.$$

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

a. Find its eigenvalues.

b. Find its determinant. Is it nonsingular?

c. Is it diagonalizable? If so, find a nonsingular matrix X such that $X^{-1}AX$ is diagonal.

REQUIRED KNOWLEDGE: **Eigenvalues, characteristic polynomial, diagonalization.**

SOLUTION:

5a: Characteristic polynomial can be found as

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} -\lambda & 1 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{bmatrix} \\ &= -\lambda[(1-\lambda)(2-\lambda) - 2] = -\lambda(\lambda^2 - 3\lambda) = -\lambda^2(\lambda - 3). \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3$.

5b: The determinant of a matrix is equal to the product of its eigenvalues. As such, it is zero. Hence, the matrix is singular.

5c: The matrix A is diagonalizable if and only if it has 3 linearly independent eigenvectors.

For $\lambda_1 = \lambda_2 = 0$, the eigenvectors can be found by solving the following homogeneous equations:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} x = 0.$$

This would yield, for instance, the following two linearly independent eigenvectors:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = 3$, we need to solve the equation:

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} x = 0.$$

This would result in, for instance,

$$x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

As we have found three linearly independent eigenvectors, the matrix is diagonalizable. A diagonalizer X can be given as

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let A, B, and C be $n \times n$ matrices. Denote the $n \times n$ zero matrix by 0_n. Consider the matrix

$$M = \begin{bmatrix} A & B \\ 0_n & C \end{bmatrix}.$$

a. Show that M is nonsingular if both A and C are nonsingular.

b. Show that

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0_n & C^{-1} \end{bmatrix}.$$

Required Knowledge: **Nonsingularity and partitioned matrices.**

Solution:

6a: Let z be such that $Mz = 0$. Partition z as

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y are n-vectors. Now, we have

$$0 = Mz = \begin{bmatrix} A & B \\ 0_n & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cy \end{bmatrix}.$$

It follows from $Cy = 0$ and the non singularity of C that $y = 0$. Therefore, we have $Ax + By = Ax = 0$. Since A is nonsingular, this can happen only if $x = 0$. Since both x and y are zero, we get $z = 0$. Therefore, M is nonsingular.

6b: Note that

$$\begin{bmatrix} A & B \\ 0_n & C \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0_n & C^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} + B0_n & -AA^{-1}BC^{-1} + BC^{-1} \\ 0_nA^{-1} + C0_n & -0_nA^{-1}BC^{-1} + CC^{-1} \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix} = I_{2n}.$$

where I_n is the $n \times n$ identity matrix. Therefore, we can conclude that

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0_n & C^{-1} \end{bmatrix}.$$